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Periodic Gibbs States of Ferromagnetic Spin Systems

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We give a complete description of the set of periodic Gibbs states at low temperatures for classical spin systems with arbitrary ferromagnetic, finite-range, interactions and fairly general even single-spin distribution of compact support on R. This extends results of Holsztynski and Slawny for the spin-1/2 case. The extension is based on recent ferromagnetic inequalities and low-temperature expansions.

KEY WORDS: Low-temperature phases; periodic Gibbs states; ferromagnetic spins.

1. INTRODUCTION

In a recent paper Holsztynski and Slawny[1] have given a complete description of all low-temperatures periodic Gibbs states for Ising spin-1/2 systems with ferromagnetic finite-range interactions on a lattice $\mathbf{Z}^d$. Their results can be summarized as follows: for spin-1/2 Ising ferromagnets, the set $\mathfrak{B}^+ = \{A \subset \mathbf{Z}^d,\ A \text{ finite} \mid \rho^+(S_A) \neq 0\}$, where $S_A = \prod_{i\in A} S_i$ and ρ^+ is the infinite-volume Gibbs state obtained with + boundary conditions, determines the set Δ_p of all periodic (and quasiperiodic) Gibbs states at sufficiently low temperatures. Using correlation inequalities, this is actually true whenever the pressure (or free energy) is differentiable with respect to the temperature, i.e., except possibly on a countable set of values of T.[2,3]

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The set $\mathfrak{B}^+$ was then shown to be determinable by algebraic means, leading, for finite-range interactions, to a complete description of Δ_p.

It is the purpose of this note to show that this result can be extended to general continuous or discrete Ising spins with even single-spin measure of compact support. Our analysis is based on (a) showing that here, too, $\mathfrak{B}^+$ determines the set Δ_p for almost all temperatures and (b) reducing the determination of $\mathfrak{B}^+$ to that of an equivalent spin-1/2 system.

Both parts involve the use of ferromagnetic inequalities. In the Appendix we reproduce the proof of one of them, namely Wells' inequality.[4]

Part (a) can be strengthened for a large class of a priori measures to all sufficiently low temperatures by the use of low-temperature expansions for the free energy, as in the case of spin-1/2 systems.

Some of the ideas used in this note were already noted by Slawny in Ref. 5, Remark 6-1, but the inequalities used here were not available then.

2. THE MAIN RESULTS

2.1. Gibbs States[6,7]

Let L be a discrete $\mathbf{Z}^d$-invariant subset of $\mathbf{R}^d$. For each $i \in \mathrm{L}$, we have a copy (K_i, ν_i) of the interval $[-1, +1]$ and of a Borel probability measure ν on $[-1, +1]$. For $\Lambda \subseteq \mathrm{L}$, we let $K_\Lambda = \prod_{i\in\Lambda} K_i, \nu_\Lambda = \prod_{i\in\Lambda}\nu_i$.

The set of *multiplicity functions* (m.f.) M is the set of all maps from L into N equal to zero except on a finite set. For $A \in M$, and $\Lambda \subset \mathrm{L}$, $A \subset \Lambda$ means $A(i) = 0$ for $i \notin \Lambda$, while $A \cap \Lambda \neq \emptyset$ means $A(i) \neq 0$ for some $i \in \Lambda$,

$$|A| = \sum_i A(i), \qquad \bar{A} = \{ i \in \mathrm{L} \,|\, A(i) \text{ is odd} \}$$

Given $A \in M$ and a family $(f_i)_{i\in\mathrm{L}}$ of functions from K_i into R, we let $f_A = \prod_i f_i^{A(i)}$

An *interaction* J is a map from M into R, $\operatorname{supp} J = \{A \in M \,|\, J(A) \neq 0\}$. A *fundamental* family for J is a set $\mathfrak{B}_0 \subset M$ such that any $A \in \operatorname{supp} J$ is the translate of exactly one A in $\mathfrak{B}_0$ (with the natural action of $\mathbf{Z}^d$ on $M, A \to A + i$). We only consider interactions having a finite fundamental family and which are translation invariant: $J(A) = J(A + i)$.

Given any finite $\Lambda \subset \mathrm{L}$ and any configuration $S_{\Lambda^c} = (S_i)_{i\in\Lambda^c}$, called a *boundary condition* (b.c.), one defines the *Hamiltonian* H_{Λ, S^c_Λ} as a function on K_Λ:

$$H_{\Lambda, S^c_\Lambda} = - \sum_{A \cap \Lambda \neq \emptyset} J(A) S_A \tag{1}$$

The *Gibbs measure* under the boundary condition S_{Λ^c} is

$$d\mu_{\Lambda,S_{\Lambda^c}} = Z^{-1}_{\Lambda,S_{\Lambda^c}} \exp(-\beta H_{\Lambda,S_{\Lambda^c}})\, d\nu_\Lambda \tag{2}$$

$$Z_{\Lambda,S_{\Lambda^c}} = \int_{K_\Lambda} \exp(-\beta H_{\Lambda,S_{\Lambda^c}})\, d\nu_\Lambda \tag{3}$$

and $\beta = 1/kT$, where T is the temperature. We let $\langle \cdots \rangle_{\Lambda,S_{\Lambda^c}}$ denote the expectation value with respect to $\mu_{\Lambda,S_{\Lambda^c}}$.

Given an inverse temperature β and an interaction J, a state ρ on K_{L} is a *Gibbs state* for (β,J) if $\forall \Lambda \subset \mathsf{L}$, Λ finite,

$$\rho_\Lambda(f) = \int_{K_{\Lambda^c}} \langle f \rangle_{\Lambda,S_{\Lambda^c}}\, d\rho_{\Lambda^c} \tag{4}$$

where ρ_Λ = restriction of ρ to K_Λ. We have $\Delta(\beta,J)$ as the set of Gibbs states for (β,J), and $\Delta_I(\beta,J)$ [resp. $\Delta_p(\beta,J)$] as the set of Gibbs states which are invariant under the natural action of $\mathbf{Z}^d$ (resp. periodic, i.e., invariant under a subgroup of finite index of $\mathbf{Z}^d$).

The pressure is defined as $\psi(\beta,J) = \lim_{\Lambda\to\infty}(1/|\Lambda|)\log Z_{\Lambda,S_{\Lambda^c}}$, which exists, is independent of the b.c., and is convex in β. The values of β for which $d\psi/d\beta$ exists are called *regular*. Since $\psi(\beta)$ is convex, the set of irregular points is at most countable. If β_0 is regular,

$$\left.\frac{d\psi}{d\beta}\right|_{\beta_0} = \rho\left[\sum_{A\in\mathfrak{B}_0} J(A)S_A\right] \tag{5}$$

for all $\rho \in \Delta_I(\beta_0,J)$ where $\mathfrak{B}_0$ is a fundamental family for J.

2.2. The State ρ^+

From now on we shall restrict ourselves to *ferromagnetic* interactions: $J(A) \geqslant 0$, $\forall A \in M$, and *even* measures ν on $[-1,+1]$, with ν not concentrated at 0.

Under these conditions, it is well known (see, e.g., Ref. 8) that the "+b.c." $S_i = +1$, $\forall i \notin A$, have the following property: $\lim_{\Lambda\to\mathsf{L}}\mu_{\Lambda,+} = \rho^+$ exists, $\rho^+ \in \Delta_I(\beta,J)$ is extremal in $\Delta(\beta,J)$, and, moreover,

$$\forall A \in M,\quad \forall \rho \in \Delta(\beta,J),\quad \rho^+(S_A) \geqslant \rho(S_A) \tag{6}$$

The group $G = \{-1,+1\}^{\mathsf{L}}$ acts by pointwise multiplication on K_{L}.

The symmetry group of J is $\mathfrak{S} = \{g \in G \mid S_A \circ g = S_A \forall A \in \operatorname{supp} J\}$.

The isotropy subgroup of ρ^+ is $\mathfrak{S}^+ = \{g \in \mathfrak{S} \mid \rho^+(S_A \circ g) = \rho^+(S_A), \forall A \in M\}$.

We also consider the group $\mathcal{P}_f(\mathsf{L})$ of the finite subsets of L, equipped with the symmetric difference, denoted $\triangle$. The subgroup $\overline{\mathfrak{B}}$ is the subgroup of $\mathcal{P}_f(\mathsf{L})$ generated by $\{\overline{A} \mid A \in \operatorname{supp} J\}$, and $\overline{\mathfrak{B}}^+$ is the subgroup of $\mathcal{P}_f(\mathsf{L})$

given by the set of $\bar{A}$ for A such that $\rho^+(S_A) \neq 0$. This is a subgroup, because, by Griffiths' inequalities,[8] $\rho^+(S_{A+B}) \geqslant \rho^+(S_A)\rho^+(S_B)$, where $A + B$ is defined pointwise and $\overline{A+B} = \bar{A} \triangle \bar{B}$. Moreover, $\mathfrak{B} \subset \mathfrak{B}^+$ because, also by Griffiths' inequalities,[8] $\rho^+(S_A) \neq 0$ if $J(A) \neq 0$.

If we identify $\mathcal{P}_f(\mathsf{L})$ as the dual, in the sense of compact Abelian groups, of $\{-1, +1)^{\mathsf{L}}$, we have the following result:

Lemma 1.[5] $\mathbb{S}/\mathbb{S}^+$ can be identified with the dual of $(\mathfrak{B}^+/\mathfrak{B})$.

We introduce the set of Gibbs states: $\Delta^+(\beta,J) = \{\rho \in \Delta(\beta,J) \,|\, \rho(S_A) = \rho^+(S_A)$ for all A such that $\bar{A} \in \mathfrak{B}\}$. The elements of G act on states by transposition. If $g \in \mathbb{S}$, $\rho_g^+ \in \Delta_+(\beta,J)$; in the converse direction, one has the following:

Proposition. $\Delta^+(\beta,J)$ is a closed, convex subset of $\Delta(\beta,J)$ in the weak* sense. All extremal states of $\Delta^+(\beta,J)$ are of the form ρ_g^+ for $g \in \mathbb{S}/\mathbb{S}^+$; therefore all states of $\Delta^+(\beta,J)$ are of the form

$$\rho = \int_{\mathbb{S}/\mathbb{S}^+} \rho_g^+ \, d\lambda(g) \tag{7}$$

for some probability measure λ on $\mathbb{S}/\mathbb{S}^+$. We have that ρ is invariant (or periodic, or ergodic) iff λ is.

For a proof, see, e.g., Ref. 9.

We may now state the main results (Theorem and Corollary below). For $a \in [0, 1]$ we let δ_a be the probability measure concentrated with equal weight on $\pm a$.

Theorem. For any translation-invariant ferromagnetic interaction J and any even measure ν on $[-1, +1]$ not concentrated at zero, we have that (a) if β_0 is regular, $\Delta_p(\beta_0,J) \subseteq \Delta^+(\beta_0,J)$, i.e., any $\rho \in \Delta_p(\beta_0,J)$ is of the form (7) with λ periodic; (b) if $\mathsf{L} = \mathsf{Z}^d$, $\exists \beta$ such that for all $\beta' \geqslant \beta$, the group $\mathfrak{B}^+$ for β', J, and ν coincides with the one of β', J, and δ_l.

Remark. The Theorem holds for any ν even and of compact support, because the restriction to $[-1, +1]$ is only a change of scale which does not affect the results, as can be seen from the proof.

Using the low-temperature expansions of Ref. 10, Theorem 2, one shows that, for suitable ν, $\psi(\beta)$ is analytic in β for β large enough. In particular, $\psi(\beta)$ is differentiable, i.e., all values of β are regular for β large; combining this, the Theorem, and Lemma 1 together with the main Theorem of Ref. 1 gives the following:

Corollary. Let $\mathsf{L} = \mathsf{Z}^d$. For J and ν as in the Theorem with the additional assumption that either $\nu(\{1\}) \neq 0$, or that $\exists \eta > 0$, and $a, b, n < \infty$, such that on $[1 - \eta, 1]$, ν is absolutely continuous with respect to the

Lebesque measure and $d\nu(s)/ds = f(s)$ satisfies

$$b < f(s)/(1-s)^n < a, \qquad s \in [1-\eta, 1]$$

then there exists a β such that for all $\beta' > \beta$ all periodic Gibbs states are of the form (7) with λ periodic and $\mathbb{S}/\mathbb{S}^+$ given by Lemma 1 and $\overline{\mathfrak{B}}^+ =$ the subgroup of $\mathcal{P}_f(\mathbb{Z}^d)$ generated by the translates of D, D being the greatest common divisor of $\overline{\mathfrak{B}}$, in the sense of Ref. 1.

Remark. With the results of Ref. 11 one may extend these results to arbitrary L. The only result which is needed is that, for $\nu = \delta_1$, $\overline{\mathfrak{B}}^+$ be independent of the relative values of the $J(A)$'s for β large.

3. PROOF OF THE THEOREM

We start with the following Lemma, whose proof is in Ref. 1, Appendix B:

Lemma 2. Under the hypotheses of the Theorem, if $\Delta_I(\beta,J) \subseteq \Delta_+(\beta,J)$, then $\Delta_p(\beta,J) \subseteq \Delta_+(\beta,J)$.

The proof of (a) follows closely the proof of Theorem 7 in Ref. 3, using the inequality (2.5) and the proof of Corollary 5′ in Ref. 12, which extends the results of Ref. 2 to continous spins.

Since $(d\psi/d\beta)|_{\beta=\beta_0}$ exists we have, using (5), (6), and the positivity of $J(A)$,

$$\rho^+(S_A) = \rho(S_A) \neq 0 \qquad \forall A \in \mathfrak{B}_0, \quad \forall \rho \in \Delta_I(\beta_0, J) \tag{8}$$

The fact that $\rho^+(S_A) \neq 0$ follows from Griffiths' inequalities[8] and (8) extends to all $A \in \operatorname{supp} J$ by translation invariance.

We want to show (8) for all A such that $\overline{A} \in \overline{\mathfrak{B}}$ and all $\rho \in \Delta_I(\beta_0,J)$. Then the conclusion will follow from the definition of $\Delta^+(\beta_0,J)$ and from Lemma 2. To this end, we use inequality (2.5) of Ref. 12, which implies: for any A, $B \in M$, and $\rho \in \Delta(\beta,J)$ and any two families (f_i), (g_i) of functions from $[-1,+1]$ into R, where each f_i is odd and monotone increasing and each g_i is odd or even with $|g_i(S_i)| \leqslant 1$, we have

$$\rho^+(f_A) - \rho(f_A) \geqslant |\rho^+(f_A g_B)\rho(g_B) - \rho^+(g_B)\rho(f_A g_B)| \tag{9}$$

We introduce the functions:

$$\sigma_i(S_i,\lambda_i) = \begin{cases} S_i & \text{if} \quad |S_i| \leqslant \lambda_i \\ \lambda_i \operatorname{sgn} S_i & \text{if} \quad |S_i| \geqslant \lambda_i \end{cases}$$

Notice that $\sigma_i(S_i,\lambda_i) = S_i$ if $\lambda_i = 1$ (since $|S_i| \leqslant 1$) and $[\sigma_i(S_i,\lambda_i)]^{2-}/\lambda_i$ converges to 1 as $\lambda_i \to 0$ except for $S_i = 0$. Moreover, $S_i - \sigma_i(S_i,\lambda_i), \sigma_i(S_i,\lambda_i)$ are odd monotone increasing functions of S_i. From inequality (9) it follows that $\rho^+(f_A) \geqslant \rho(f_A)$ for any $\rho \in \Delta(\beta_0,J)$, where $f_i = S_i - \sigma_i(S_i,\lambda_i)$ or $\sigma_i(S_i,\lambda_i)$. Therefore, expanding $S_A = \prod_{i'} (S_i - \sigma_i + \sigma_i)^{A(i)}$, we get that $\rho^+(S_A)$

$= \rho(S_A)$ implies

$$\rho^+(\sigma_A) = \rho(\sigma_A) \tag{10}$$

Also, by (9) we have that $\rho^+(\sigma_A) = \rho(\sigma_A)$ and $\rho^+(\sigma_B) = \rho(\sigma_B) \neq 0$ imply

$$\rho^+(\sigma_A \sigma_B) = \rho(\sigma_A \sigma_B) \tag{11}$$

Using (8) and (10), we have that $\rho^+(\sigma_A) = \rho(\sigma_A)$ $\forall A \in \operatorname{supp} J, \forall \rho \in \Delta_I(\beta_0, J)$, and by choosing suitable λ_i, dividing by λ_i, and letting some $\lambda_i = 1$ and others tend to zero, we have

$$\rho^+(S_{\bar{A}}) = \rho(S_{\bar{A}}), \qquad \text{where} \quad S_{\bar{A}} = \prod_{i \in \bar{A}} S_i \tag{12}$$

But by (11), $\rho^+(\sigma_A^2) = \rho(\sigma_A^2)$, and using suitable λ_i, we deduce $\rho^+(S_i^2) = \rho(S_i^2)$ $\forall i$ such that $A(i) \neq 0$. For any $i \in \mathsf{L}$, either $A(i) \neq 0$ for some $A \in \operatorname{supp} J$ or $\rho^+(S_i^2) = \rho(S_i^2)$ holds trivially. Combining this with (12) and (11), we have

$$\rho^+\left(S_{\bar{A}} \prod_i S_i^{2n_i}\right) = \rho\left(S_{\bar{A}} \prod_i S_i^{2n_i}\right) \tag{13}$$

where the n_i are positive integers. Equation (13) shows that ρ^+ and ρ coincide on all $B \in M$ such that $\bar{B} = \bar{A}$.

Given (13), we finish the proof by induction: (8) and (11) with all $\lambda_i = 1$ generate the coincidence of ρ and ρ^+ on all functions S_{A+B}, with $\rho^+(S_A) = \rho(S_A)$, $B \in \operatorname{supp} J$, and we use $\overline{A + B} = \bar{A} \triangle \bar{B}$.

The proof of (b) follows from the following two inequalities: For any ν even $\neq \delta_0$ and with $\operatorname{supp} \nu \subset [-1, +1]$, $\exists a > 0$ such that $\forall A \in M$,

$$\langle S_A \rangle_{a^2,+} \leqslant \langle S_A \rangle_{\nu,+} \leqslant \langle S_A \rangle_{1,+} \tag{14}$$

where on the lhs and the rhs we have replaced the measure $\nu_i(S_i)$ at each site by $\delta_{a^2}(S_i)$ or $\delta_1(S_i)$. All expectation values are taken with +b.c. and in any $\Lambda \subseteq \mathsf{L}$. The first inequality is due to Wells[(4)] and its proof is in the Appendix. The second one follows from Griffiths' inequalities:[(7)] we write $S_i = \sigma_i r_i$, where $\sigma_i = \pm 1$ and $r_i \in [0, 1]$. By conditioning on the values of r_i, $i \in \Lambda$,

$$\langle S_A \rangle_{\nu,+} = \int_0^1 r_A \langle \sigma_A \rangle_+ (Jr)\, d\varphi(r)$$

where φ is a probability measure on $[0, 1]^{|\Lambda|}$ and $(Jr)(A) = J(A) r_A$. But, by Griffiths' inequalities,

$$\langle \sigma_A \rangle_+ (Jr) \leqslant \langle \sigma_A \rangle_+ (J) = \langle \sigma_A \rangle_{1,+}$$

because $r_i \leqslant 1$. Also, $\int_0^1 r_A \, d\varphi(r) \leqslant 1$, which shows (14).

By scaling, $\langle S_A \rangle_{a,+} = a^{|A|} \langle S_A \rangle_{1,+}$ with interactions $J'(A) = J(A) a^{|A|}$. But for β large enough (depending on J and a, i.e., on J and ν), $\mathfrak{B}^+$ is the same for J and for J' (see Ref. 1). Hence the conclusion of (b) in the Theorem follows from (14).

Remark. 1. If we have "unbounded spins," i.e., the measure ν_i is not of compact support but is suitably decaying at infinity, then the first inequality in (14) holds. This gives an easy way to prove the existence of phase transitions for such models, as was noted by Wells.[4] Similar arguments were given earlier by Nelson[13] and van Beijeren and Sylvester.[14]

2. In Ref. 1, Appendix B, Lemma 2 is extended to "quasiperiodic" Gibbs states and all our results extend to this class.

3. There is an interesting example which shows why the restriction to ferromagnetic interactions is rather subtle. Let

$$H = J \sum_{\langle ij \rangle \in \Lambda} (S_i - S_j)^2 - \mu \sum_{i \in \Lambda} S_i^2$$

where the first sum is over nearest neighbors, and ν is concentrated on $-1, 0, +1$ with equal weights. Then (see Ref. 15 or Ref. 16) one has for every β large a value of μ such that there are three extremal translation-invariant Gibbs states. If we write $(S_i - S_j)^2 = S_i^2 + S_j^2 - 2S_iS_j$, we see that our Hamiltonian is not ferromagnetic, because of the S_i^2 term. However, we may absorb the terms in S_i^2 (and μS_i^2) into the single-spin measure and the Hamiltonian would then be ferromagnetic, but the phases (3) do not correspond to the spin-1/2 case. The reason is that we have put β-dependent terms in the single-spin measure and this is different from the situation considered in this note. Actually, if we put $\mu = \mu(\beta)$ so that we have three phases, absorb all the S_i^2 terms in the single-spin measure, and put a new parameter β' multiplying the term $\sum_{\langle ij \rangle} S_iS_j$, then $d\psi/d\beta'$ does not exist for the value of β' for which there are three phases. Indeed, if $d\psi/d\beta'$ existed, then the proof of part (a) in the Theorem would show that all translation-invariant Gibbs states coincide on S_i^2, which is known to be false in this case.[15,16] So, in this way we have constructed an example of a ferromagnetic (spin 1) system where some value of the temperature is not regular.

APPENDIX. PROOF OF WELLS' INEQUALITY

We prove the first inequality in (14), following Ref. 4. Using duplicate variables, this amounts to showing that there exists an $a > 0$ such that

$$\int (S_A - S_A') \exp\Big[\sum_{A \subset \Lambda} J(A)(S_A + S_A') \Big] \times \prod_{i \in \Lambda} d\nu_i(S_i)\, \delta\big(S_i'^2 - a^2\big)\, dS_i' \geqslant 0$$

for all $A \in M$ and all J ferromagnetic. Expanding in the usual way, it is enough to show that for all n, $m \in \mathbf{N}$,

$$\int (S+S')^m (S-S')^n \, d\nu(S)\, \delta(S'^2 - a^2)\, dS' \geqslant 0$$

By symmetry we can assume that m, n are odd and $m \geqslant n$. Then, integrating over S' gives

$$\int (S^2 - a^2)^n \left[(S-a)^{m-n} + (S+a)^{m-n} \right] d\nu(S) \tag{A1}$$

since $m - n$ is even, the term in the brackets is an increasing function of $|S|$. Therefore, it is larger than $(2a)^{m-n}$ if $|S| \geqslant a$ and less than $(2a)^{m-n}$ if $|S| \leqslant a$. Therefore (A1) is bounded below by

$$(2a)^{m-n} \int (S^2 - a^2)^n \, d\nu(S)$$

If ν is not δ_0, there exists a $K \neq 0$ such that $\nu([K, \infty]) \neq 0$. Let $a = K/M$ where M is chosen below. We have

$$\begin{aligned}
&\int \left(S^2 - \frac{K^2}{M^2} \right)^n d\nu(S) \\
&\quad = \int_{|S| \leqslant K/M} \left(S^2 - \frac{K^2}{M^2} \right)^n d\nu(S) + \int_{K/M \leqslant |S| \leqslant K} \left(S^2 - \frac{K^2}{M^2} \right)^n d\nu(S) \\
&\qquad + \int_{|S| \geqslant K} \left(S^2 - \frac{K^2}{M^2} \right)^n d\nu(S) \\
&\quad \geqslant -\left(\frac{K^2}{M^2} \right)^n \nu\left(\left[0, \frac{K}{M} \right] \right) + 2\left(K^2 - \frac{K^2}{M^2} \right)^n \nu([K, \infty]) \\
&\quad = \left(\frac{K^2}{M^2} \right)^n \left[-\nu\left(\left[0, \frac{K}{M} \right] \right) + 2(M^2 - 1)^n \nu([K, \infty]) \right]
\end{aligned}$$

It is immediate that this last expression is positive for all n, if we choose M large enough.

Remarks. It is easy to check that Wells' proof, together with Ginibre's proof of his inequalities,[17] yields a similar inequality for rotators: let $K_i = \mathbf{R}^2$ and $\nu_i =$ a rotation-invariant measure on $\mathbf{R}^2$ (suitably decaying at infinity). We have

$$-H = \sum_{m,A} J(m, A) r_A \cos m\theta$$

where m is a m.f. with values in $\mathbf{Z}$ instead of $\mathbf{N}$ and $m\theta = \sum m(i)\theta_i$. Assume

$J(m, A) \geqslant 0$. Then there exists $a > 0$ such that $\forall B$, $\forall n$,

$$\langle r_B \cos n\theta \rangle' \leqslant \langle r_B \cos n\theta \rangle$$

where in $\langle \cdots \rangle'$ we substitute $d\nu'(r, \theta) = \delta(r - a)\, dr\, d\theta$ for $d\nu$.

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